ON THE INTEGRATION OF UNBOUNDED FUNCTIONS*

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1. *Introduction.* The author has shown† that F. Riesz' treatment of integration‡ leads in every case to the Lebesgue integral. This demonstration makes possible a complete development of the theory of Lebesgue integration from the Riesz point of view.§ Such a development offers a number of advantages over the usual treatment and is especially desirable when one wishes to build a Lebesgue theory on a previous treatment of the Riemann integral. The purpose of the present paper is to emphasize further the importance of the Riesz point of view in a treatment of the general subject of integration by establishing additional relations between sequences of simple functions and functions that are summable in the senses of Lebesgue, Harnack, Denjoy, Denjoy-Khintchine-Young, and Young. The terminology and notation of Riesz' paper|| are used.

2. *Preliminary Definitions and Theorems.* In this section we give a number of definitions and theorems which are well known but which are essential for the development of later theorems.

Simple function.¶ A function $\phi(x)$ is said to be a simple function on X: $a \leqq x \leqq b$, if there exist $n+1$ points: $x_0 = a < x_1 < x_2 < \cdots < x_n = b$, such that $\phi(x) \equiv \phi_i$, a constant, on $I_i: x_i < x < x_{i+1}$.

Null set. A set of points K is said to be of measure zero if for each $\epsilon > 0$, there exists an at most countably infinite set of intervals such that each point of K is an interior point of some one of these intervals and the sum of the lengths of the intervals

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† This Bulletin, vol. 37(1931), pp. 561–564.

‡ Acta Mathematica, vol. 42(1920), pp. 191–205.

§ This point of view is essentially that used in the ordinary treatment of the Riemann integral.

|| Loc. cit. We work entirely in the real domain.

¶ These functions have also been called *horizontal* or *step functions.* In this connection see Ettlinger, American Journal of Mathematics, vol. 48 (1926), pp. 215–222.