# NOTE ON VOLTERRA AND FREDHOLM PRODUCTS OF SYMMETRIC KERNELS*

BY L. M. BLUMENTHAL

1. *Introduction.*† The purpose of this paper is to establish two theorems concerning the Volterra and the Fredholm products of two continuous, symmetric functions. These two theorems were obtained in the course of an investigation that had as its object the determination of the existence of permutable functions (first kind) of the second order; that is, the determination of functions $K_1(x, y)$, $K_2(x, y)$ such that

$$K_1K_2K_2K_1 = K_2K_1K_1K_2, \tag{1}$$

but

$$K_1K_2 \neq K_2K_1,$$

where

$$K_1K_2 = \int_x^y K_1(x, t)K_2(t, y)dt.$$

To satisfy (1),‡ it is sufficient to find two functions $K_1$, $K_2$ such that

$$K_1K_2 = -K_2K_1, \tag{2}$$

for, composing (on the right) the left-hand member with $K_2K_1$, and the right-hand member with its equal, $-K_1K_2$, we obtain (1). If two functions satisfy (2), we shall call them *skew permutable*. Assume, now, that $K_1(x, y)$, $K_2(x, y)$ are symmetric functions of their arguments. We have the following lemma.

LEMMA. *The Volterra product of two symmetric functions is itself a symmetric function if and only if the two functions are skew permutable.*

Let

$$K(x, y) = \int_x^y K_1(x, \xi)K_2(\xi, y)d\xi = K_1K_2;$$

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‡ In this part of the paper we are concerned entirely with Volterra composition.