ON THE APPLICATION OF MARKOFF'S THEOREM TO PROBLEMS OF APPROXIMATION IN THE COMPLEX DOMAIN*

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In a recent note,† the writer has discussed an extension to the complex domain of a method previously used in connection with problems of the approximate representation of real functions, in which the proof of convergence is based on Bernstein's theorem on the derivative of a polynomial or trigonometric sum. The discussion for the case of a complex variable involved certain restrictions on the boundary of the region with which the problem was concerned. The object of the present paper is to show how these restrictions may be somewhat lightened, at the expense, to be sure, of a compensating increase in the stringency of the hypotheses on the function to be approximated, if Markoff's theorem on the derivative of a polynomial is used in place of that of Bernstein.

Markoff's theorem may be stated for the purpose in hand as follows.

If $P_n(z)$ is a polynomial of the nth degree such that $|P_n(z)| \leqq L$ at all points of a line segment of length $2h$ in the z-plane, then $|P_n'(z)| \leqq n^2L/h$ at all points of the same segment.

The theorem is commonly stated for an interval of the axis of reals, and more particularly for the interval $(-1, 1)$, thus:‡ If $|P_n(x)| \leqq L$ for $-1 \leqq x \leqq 1$, then $|P_n'(x)| \leqq n^2L$ throughout the interval. If the hypothesis of the more general statement is given for the segment connecting the points z_1 and z_2, the substitution $z = \alpha + \beta z'$, with $\alpha = \frac{1}{2}(z_1 + z_2)$, $\beta = \frac{1}{2}(z_2 - z_1)$, $|\beta| = h$, reduces one to the other. In the formulation for the interval $(-1, 1)$, to be sure, the coefficients are ordinarily thought of as

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† *On certain problems of approximation in the complex domain*, this Bulletin vol. 36 (1930), pp. 851–857.

‡ See, for example, Marcel Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; pp. 359–360.