TWO BOOKS ON FOUNDATIONS OF GEOMETRY

Grundlagen der Geometrie. By Gerhard Hessenberg. Edited by W. Schwan. Berlin, de Gruyter (Göschens Lehrbücherei, 1. Gruppe, Band 17), 1930. 143 pp., 77 figs.

Vorlesungen über Grundlagen der Geometrie. By Kurt Reidemeister. Berlin, Springer (Die Grundlehren der mathematischen Wissenschaften, vol. 32), 1930. x+147 pp., 37 figs.

It is well that the simultaneous publication of two books on the bases of geometry, both by most competent authors, both making noteworthy contributions to the subject, should remind us of the progress which still continues after so many centuries. In writing on this topic, an author chooses between the exclusive use of strict logic, on the one hand, and, on the other, enough of an appeal to the pictorial sense to illuminate the subject and make it more accessible to readers with but a moderate power of abstraction. Reidemeister's book is of the first type; while that which was begun by Hessenberg and, after his death, completed by Schwan, is rather of the other kind.

The latter book, to be sure, almost completely satisfies a logician's demands, but there are a few diverting gaps in the argument—probably the result of divided authorship. Thus (p. 33), when the nature of lines and their segments has been carefully discussed, parallelism and direction are introduced with no foundation, no mention that there is such a thing as a plane. On page 77 we find the first use of the word "motion,"—"Let a plane E, which coincides with A, be so moved . . . ," and there is no hint of the concept of motion or its relation to congruence.

Distinctly good is the discussion of the logical relations between the Pascal and Desargues theorems, axioms of continuity (particularly that of Archimedes), order, incidence and parallelism, and the fundamental theorems of plane and solid projective geometry. There is, of course, Hessenberg's proof that Desargues' Theorem is a consequence of Pascal's. Important "artificial" geometries illustrate certain independences. Thus we have Moulton's non-desarguean plane geometry and various non-archimedean geometries. The latter group of examples shows how the axioms of incidence, order and parallelism are insufficient to prove Pascal's Theorem, and how the axioms named, plus this theorem, are still not enough to establish the axiom of Archimedes. Since Pascal's Theorem, or its corollary, Desargues', is sufficient for the transition from two dimensions to more, the problem is proposed of discovering what axiom, weaker than the archimedean, will suffice to establish this transition without the use of congruence.

After a short, clear initial chapter on the concepts of equality, order, and continuity, that of congruence leads up to a discussion of numerical measurement. Hessenberg treats measurement of segments, Schwan continues with that of vectors—with the gap in definitions already noted. In the measurement of angles, Schwan distinguishes between the angles themselves (figures consisting of two half-lines and, in measure, never exceeding π) and angular fields