USEFUL FUNCTIONS ASSOCIATED WITH RATIONAL CUBIC CURVES*

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1. *Introduction.* The usual method of determining the Plücker numbers for plane curves is generally laborious. However, in certain cases, by the use of special theorems, the numbers may be determined at once. Thus, the curve $x_1:x_2:x_3=f(t):\phi(t):\psi(t)$, where f, ϕ, ψ are polynomials of degree 3 in t and the coefficients of t^2 in each are zero, has one cusp.† Also any cusp or node at a vertex of the triangle of reference is easily recognized by the form of the equation. The purpose of this paper is to derive some of the properties of two functions that are useful in determining whether the rational cubic is nodal or cuspidal.

2. *The Cubic Circumscribed about the Triangle of Reference.* The parametric equations of the cubic are

$$\text{(1)} \qquad \rho x_i = (\lambda - \lambda_i)(\lambda - \lambda_{i+2})(\lambda - s_i), \qquad (i = 1, 2, 3),$$

where λ_i, λ_{i+2}, s_i are the points of intersection of the cubic with the side x_i of the triangle, λ_i and λ_{i+2} being vertices. Let A represent the function

$$(\lambda_3 - s_1)(\lambda_1 - s_2)(\lambda_2 - s_3) - (\lambda_1 - s_1)(\lambda_2 - s_2)(\lambda_3 - s_3).$$

THEOREM 1. *The cubic* (1) *has a cusp at one of the vertices of the triangle of reference or is nodal when A vanishes.*

PROOF. The class of the cubic is given by the degree of λ in the equation of a tangent line to the curve from any point (x_1, x_2, x_3), after all common factors are eliminated. The tangent line is given by the equation

$$\text{(2)} \qquad \sum_{i=1}^{3} x_i(a_i\lambda^4 + b_i\lambda^3 + c_i\lambda^2 + d_i\lambda + e_i) = 0,$$

where

$$a_i = -\lambda_i + \lambda_{i+2} + s_{i+2} - s_{i+1},$$
$$e_i = \lambda_{i+1}^2(\lambda_i\lambda_{i+2}s_{i+2} + \lambda_{i+2}s_{i+1}s_{i+2} - \lambda_i s_{i+1}s_{i+2} - \lambda_i\lambda_{i+2}s_{i+1}),$$

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† For the general theorem, see Hilton, *Plane Algebraic Curves*, 1920, p. 151.