LEFSCHETZ ON TOPOLOGY

Topology. By Solomon Lefschetz. New York (American Mathematical Society Colloquium Publications, Volume 12), published by this Society, 1930. 409 pp.

The rapid growth of interest in the science of analysis situs has been one of the striking facts of recent mathematical history. Of decided importance in the development of that branch of the science which deals with n-dimensional manifolds was the publication in 1922 of Veblen's Colloquium Lectures. In this volume the connectivity theory, one of the finest creations of Poincaré, was put onto the rigorous foundations which had been prepared for it in the work of Veblen and Alexander, and the many problems which the more exacting viewpoint entailed were clearly brought into evidence. The "speedy obsolescence" which Veblen wished for his book has by no means taken place; but the success of his work is evident from the many important advances to which it has given stimulus. One of the outstanding leaders of this recent progress is the author of the present volume. His researches have extended over a period of several years, and have now been brought to a brilliant culmination in this latest addition to the Colloquium Publication series.

Almost the whole of Lefschetz's work in analysis situs has centered around the following fundamental problem: a given space is subjected to a continuous transformation into itself; how can a topologically significant index be attached to the totality of fixed points and how can its value be computed in terms of the invariants of the situation? In an early series of papers on surface transformations, Brouwer included the first explicit solution of this problem for a very special case. If a sphere is subjected to a single-valued sense-preserving transformation into itself, then, as Brouwer pointed out, the algebraic sum of the multiplicities of the fixed points is always two, and from this the existence of at least one fixed point is immediately inferred. Other authors treated somewhat more general cases and a number of interesting results were obtained, notably by Birkhoff and Alexander. It was Lefschetz, however, who in 1923 proposed a method of attack which penetrated into the heart of the problem in its most general form, and brought it immediately within working range of the tools of n-dimensional analysis situs as they had been developed in Veblen's lectures. If M' is a copy of a given space M, let us form, said Lefschetz, the product space $M\times M'$. Then a transformation T of M may be thought of as a pairing off of the points of M and M', and the totality of point-pairs will constitute a subspace V of $M\times M'$. The identity in particular will lead to subspace V_0, and the fixed points of T will then correspond to the intersections of the two subspaces V and V_0. Thus the concept of transformation, with its connotation of change, is essentially a static geometric situation, and as such can most advantageously be studied.

This, then, is the guiding principle for Lefschetz's attack on the problem, and it is interesting to observe the fruitfulness of an idea of such simplicity and intuitive content. There is hardly an important situation which does not yield