A REMARK CONCERNING THE NECESSARY CONDITION OF WEIERSTRASS*

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Let us consider a class $\Re$ of rectifiable curves C lying in a point set A of n-dimensional space, and an integral $F(C) = \int_C f(x, x')ds$, where $x = (x^1, \cdots, x^n)$ and s connotes that we use the length of arc as parameter. Suppose that a certain curve $C: x = x(s)$ minimizes $F(C)$ in $\Re$, and denote by L the set of points of C which are interior to A and of indifference with respect to $\Re$ and A. Then for almost all points of L we have‡ $E(x(s), x'(s), \bar{x}') \geqq 0$ for all sets of numbers $\bar{x}'$. Given now a particular point $x(s_0)$ of L; when can we say that the inequality holds at $x(s_0)$?

It has already been shown§ that the inequality holds if $x'(s_0)$ exists, $\Sigma [x^{i'}(s_0)]^2 > 0$, and the $x^{i'}(s)$ are all approximately continuous at s_0. We will now show that the inequality also holds if $\Sigma(x^{i'}(s_0))^2 = 1$. (As is well known, this sum never exceeds 1, and is equal to 1 almost everywhere.)

Suppose then that $\Sigma [x^{i'}(s_0)]^2 = 1$ and that in contradiction to our statement there exists an $\bar{x}'$ such that $E(x(s_0), x'(s_0), \bar{x}') = -2k < 0$. Denote by $\alpha(s)$ the angle between $x'(s)$ and $x'(s_0)$. The function

$$\phi(s) = \frac{d}{ds} \Big[\sum x^i(s) x^{i'}(s_0) \Big] = \sum x^{i'}(s) x^{i'}(s_0)$$

$$= \Big\{ \sum [x^{i'}(s)]^2 \Big\}^{1/2} \Big\{ \sum [x^{i'}(s_0)]^2 \Big\}^{1/2} \cos \alpha(s)$$

is defined for almost all values of s, and $|\phi(s)| \leqq |\cos \alpha(s)|$. By the continuity of E, we can find positive numbers ϵ, δ such that $E(x(s), x'(s), \bar{x}') < -k$ for all s such that $|s - s_0| \leqq \epsilon$, $\phi(s) \geqq 1 - \delta$; and if ϵ be small enough, $x(s)$ will be in L. But $\phi(s_0) = 1$ and $\phi(s)$