A RELATIONSHIP IN SPHERICAL HARMONICS

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It is known that a solid spherical harmonic about one set of axes can be expressed as a solid spherical harmonic with constant coefficients about a new set of axes whose origin is displaced from that of the first. A new relationship has been found for making this transformation and a proof is herein given.

Consider the two sets of cartesian coordinates x, y, z and x_1, y_1, z_1, whose origins O and O_1 are separated by a distance h. Let the two axes x and x_1 be coincident, but let the positive sense of the two axes be opposite. Consider the point S which has the two sets of polar coordinates defined by the equations

$$\begin{aligned} x_1 &= \rho_1 \cos\theta_1 = \rho_1\mu_1, & x &= \rho\cos\theta = \rho\mu, \\ y_1 &= \rho_1 \sin\theta_1 \sin\phi_1, & y &= \rho\sin\theta\sin\phi, \\ z_1 &= \rho_1 \sin\theta_1 \cos\phi_1, & z &= \rho\sin\theta\cos\phi. \end{aligned}$$

The transformation to be obtained may be expressed symbolically as follows:

$$(1) \qquad \sum_{m=0}^{\infty} \rho_1^{-(m+1)} Y_m(\mu_1, \phi_1) = \sum_{m=0}^{\infty} f_m(\rho) Y_m(\mu, \phi),$$

in which Y_m signifies the usual spherical harmonic. Such a transformation has been given previously by B. Datta.* The new relationship which has been found to give the transformation (1) and which appears to have been unnoticed before is

$$(2) \qquad \frac{P_m^n(\mu_1)}{\rho_1^{m+1}} = a_{(m-n)}(1-\mu^2)^{n/2} \frac{\partial^{m-n}}{\partial h^{m-n}}\left(\frac{1}{h^n}\frac{\partial^n}{\partial \mu^n}\frac{1}{\rho_1}\right),$$

in which $a_{(m-n)}$ designates the term $(-1)^{m-n}/(m-n)!$. A proof of this relationship follows.

Relative to the partial differentiation on the right side of (2), ρ_1 is to be expressed in terms of ρ, μ and h. At certain stages of the calculation it will be convenient to express ρ_1 by means of the equation

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* Tôhoku Mathematical Journal, vol. 15 (1919), p. 166.