THE LATENT ROOTS OF A MATRIX OF SPECIAL TYPE

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1. *Introduction.* In this paper the latent roots of a matrix, each of whose elements are suitably restricted rational functions of a square matrix A, are discussed. In particular a theorem is proved about the latent roots of such a matrix, which reduces in the simplest case to the familiar theorem on the value of the latent roots of a rational function of a matrix A. The third section illustrates the fact that the values of circulant determinants, and of determinants composed of blocks which are themselves circulants, can be easily deduced from their matrix properties.

2. *Latent Roots.* Let A be a square matrix of n rows and n columns and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the latent roots of A, where the λ_i need not be distinct. Then by a well known theorem on matrices* there exists a non-singular matrix X, which transforms A into a matrix whose elements in the leading diagonal are the latent roots of A, while all the elements to the left of the leading diagonal are zero. This may be expressed by the matrix equation

$$XAX^{-1} = \begin{pmatrix} \lambda_1 & r_{12} \cdots r_{1n} \\ 0 & \lambda_2 \cdots r_{2n} \\ \cdots & \cdots \\ \cdots & \cdots \\ 0 & 0 \cdots \lambda_n \end{pmatrix} = \Lambda. \tag{1}$$

It follows from (1) that, if $f(A)$ is a rational function of the matrix A, of which the denominator is non-singular, the matrix X transforms the matrix $f(A)$ into a matrix of the same type as Λ, where the elements in the leading diagonal are $f(\lambda_1)$, $f(\lambda_2), \cdots, f(\lambda_n)$. Or

$$Xf(A)X^{-1} = f(\Lambda), \tag{2}$$

* A. Wintner, *Spektraltheorie der unendlichen Matrizen*, pp. 20–22.