PROOF OF A WARING THEOREM ON FIFTH POWERS*

BY L. E. DICKSON

In 1770, E. Waring conjectured that "every number is a sum of nine cubes, also is a sum of 19 fourth powers, and so on to infinity."

By elaborate and very delicate analysis, Hardy and Littlewood obtained remarkable asymptotic results. Their proofs apply only to numbers exceeding an extremely large limit L.

I shall here explain my method to cover the numbers $<L$ and hence finally obtain universal theorems holding from 1 to infinity.

For greater clearness, I shall first consider briefly the case of cubes. Jacobi inspired the construction of tables showing the least number of positive cubes (or fourth powers) whose sum yields the tabulated number. In 1851, he published the table to 12,000 by Z. Dahse. In 1903, von Sterneck made a table to 40,000 and found not merely that nine cubes always suffice, but also the more vital fact that all numbers between 8,042 and 40,000 are sums of six cubes.

I shall prove that every number N lying between 40,000 and 67,000 is a sum of seven cubes. From all these numbers we subtract 27,000, which is the cube of 30. Take 27,000 from 40,000 and you have 13,000. Take 27,000 from 67,000 and you have 40,000. Since $N-30^3$ therefore lies between 13,000 and 40,000, it is a sum S of six cubes by the result quoted. Hence $N=30^3+S$ is a sum of seven cubes.

Similarly, every number M lying between 67,000 and 94,872 is a sum of seven cubes, since $M-38^3=M-54{,}872$ lies between 12,128 and 40,000 and hence is a sum of six cubes, so that M is a sum of seven cubes.

This argument may be repeated until we have finally subtracted the cube of 103. Our result is that all numbers between 8,042 and 1,132,727 are sums of seven cubes.

---

* An address, presented to the Society at the request of the program committee, April 4, 1931. The presentation was made with the aid of lantern slides.