DUNHAM JACKSON ON APPROXIMATION

The Theory of Approximation. By Dunham Jackson. New York (American Mathematical Society Colloquium Publications, Volume 11), published by this Society, 1930. v+178 pp.

In 1885 Weierstrass announced his now famous theorem: *Any continuous function $f(x)$, defined over a finite interval (a, b), can be approximated in (a, b) uniformly and indefinitely by a sequence of polynomials $P_n(x)$, whose degrees n tend to infinity*, that is, $\lim_{n\to\infty} P_n(x)-f(x)=0$, $(a\leqq x\leqq b)$. The same holds true, if polynomials are replaced by finite trigonometric sums of ever increasing orders. We shall speak of this, for brevity, as the (ordinary) polynomial or trigonometric approximation.

One can hardly overestimate the influence exerted by this theorem, now commonly known as "Weierstrass' Theorem," on the development of modern analysis. It suffices to say that the theorem in question very often enables the investigator to extend, at a single stroke, a property previously established *for polynomials only* to the infinitely wider class of continuous functions.

It seems, therefore, quite fitting that the *Theory of Approximation* is the subject of one of the series of the Colloquium Lectures of the Society, a subject for which no one is better qualified as lecturer than Professor Dunham Jackson, who has made to it such important contributions. Those who had the pleasure of hearing the Colloquium Lectures or other papers read before the Society by Professor Jackson always admire their lucidity and elegance of style. The present book possesses these qualities to a high degree. The exposition is clear, detailed and interesting, and in many points distinctly novel.

The book consists of five chapters. Chapter 1 is devoted to the ordinary trigonometric and polynomial approximation of continuous functions. After a brief introduction, where Weierstrass' Theorem is stated, the author proves the fundamental Theorem 1 on trigonometric approximation of a function $f(x)$ satisfying a Lipschitz condition. The proof is based upon the use of the *singular* integral

$$h_m\int_{-\pi/2}^{\pi/2} f(x+2u)F_m(u)du,\quad F_m(u)=\left(\frac{\sin mu}{m\sin u}\right)^4,\quad \frac{1}{h_m}=\int_{-\pi/2}^{\pi/2}F_m(u)du,$$

introduced by Jackson in his Thesis; the analysis is carried out in a detailed manner. This leads to Theorem 2 on trigonometric approximation of any continuous function, given its modulus of continuity $\omega(\delta)$, through an ingenious device also introduced by the author in his Thesis. A closer analysis of the approximating sum employed in Theorem 1 leads further to Theorems 3, 4 on trigonometric approximation of a function whose pth derivative satisfies a Lipschitz condition or has a given modulus of continuity. In the next section the author states similar theorems for polynomial approximation, reducing this to the previous case by means of the substitution $x=\cos\theta$. The next section is devoted to a brief discussion of the rapidity of convergence of Fourier and Legendre series, which is taken up with more details in Chapter 2. We note an extremely elegant proof (pages 27–28) of the following property of Legendre's