PROPERTIES OF THE OPERATOR $z^{-\nu} \log z$, WHERE $z = d/dx$*

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1. *Introduction.* A considerable bibliography with modern increments has been built up around the interpretation and use of the operators z^{ν} and $z^{-\nu}$, $z = d/dx$, where ν may assume fractional as well as integral values.† The peculiar efficacy of these symbols in the solution of integral equations of Volterra type on the one hand and in the resolution of difficulties in certain types of electrical transmission problems on the other has made a study of their properties of more than passing interest.

The essential peculiarity of the operators z^{ν} and $z^{-\nu}$ regarded as analytic functions is found in the fact that they possess branch points at the origin. It becomes of interest, therefore, to inquire into the existence of other operators with branch points at $z = 0$, as for example $\log z$. Wiener, in a paper which employs the Fourier transform of a function as its basis of definition, has established a rigorous foundation for the discussion of such branch point operators.‡ He illustrates his method by applying it to the operator $z^{1/2}$, but fails to give explicit consideration to $\log z$. In a subsequent paper F. Sbrana§ has supplied this deficiency by means of a method similar to that of Wiener in its use of the Fourier transform.

The original suggestion, however, is due to V. Volterra¶ who formulated it in a theory of *logarithms of composition*, which he applied effectively in the solution of the integral equation

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† See for example, J. D. Tamarkin, *On integrable solutions of Abel's integral equation*, Annals of Mathematics, vol. 31 (1930), pp. 218–229.

‡ *The operational calculus*, Mathematische Annalen, vol. 95 (1925–26), pp. 557–584.

§ *Sull'operazione infinitesimale nel gruppo delle derivazione*, Atti dei Lincei, (6), vol. 11 (1930), pp. 364–368.

¶ See V. Volterra and J. Pérès, *Leçons sur la Composition et les Fonctions Permutables*, Paris, 1924, Chap. 8. See also H. T. Davis, *The Inversion of Integrals of Volterra Type*, Indiana University, 1927, pp. 62–66.