SUMS OF FOUR OR MORE VALUES OF $\mu x^2+\nu x$ FOR INTEGERS x*

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1. *Introduction.* My object is to prove the following theorem.

THEOREM 1. *Let* $0<\nu<\mu, f(x)=\mu x^2+\nu x$. *Let T denote the table of all sums of four values of* $f(x)$ *for integers x arranged in order of magnitude. The largest gap between consecutive entries of T is*

$$(1)\qquad \mu-\nu,\ if\ \mu\geqq 3\nu/2;\qquad 5\nu-3\mu,\ if\ \mu\leqq 3\nu/2.$$

An immediate corollary is the following result.

THEOREM 2. *Let* $0<\nu<\mu$, $s\geqq 4$. *The largest gap in the table of all sums of s values of* $f(x)$ *for integers x is*

$$(2)\ \mu-\nu,\ if\ s\mu\geqq(s+2)\nu;\quad (s+1)\nu-(s-1)\mu,\ if\ s\mu\leqq(s+2)\nu.$$

For, if $s\geqq 4$, we need only add $(s-4)f(-1)$ to every entry of T, notice that a gap $\mu-\nu$ actually occurs from $4f(0)$ to $f(-1)+3f(0)$, that no gap greater than $\mu-\nu$ can exceed $5\nu-3\mu-(s-4)(\mu-\nu)$, and that the last number actually occurs, when it is positive, as the gap from $sf(-1)$ to $f(1)+(s-1)f(0)$.

Let us now recall‡ that the only quadratic functions $q(x)$ which are integers $\geqq 0$ for every integer x, and which take the values 0 and 1 for certain integers x, are obtained from the function

$$(3)\qquad \tfrac{1}{2}mx^2+\tfrac{1}{2}(m-2)x,$$

where m is a positive integer, by replacing x by $x-k$ or $k-x$, k an integer. By Theorem 2, the table of all sums of s values of $q(x)$ possesses as its maximum gap the number 1 if $3\leqq m\leqq s+2$, $m-(s+1)$ if $m\geqq s+2$. One corollary is that every integer $\geqq 0$ is a sum of $m-2$ values of (3) for integers x, all but four of which are 0 or 1, at least if $m\geqq 6$; and of four values if $m=3$, 4, 5; (previously proved by Dickson).

* Presented to the Society, November 29, 1930.

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‡ L. E. Dickson, this Bulletin, vol. 33 (1927), p. 714.