ON THE EXTENSION OF THE GAUSS MEAN-VALUE THEOREM TO CIRCLES IN THE NEIGHBORHOOD OF ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS

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1. *Introduction.* Let $f(x, y)$ be a function harmonic in a plane region R except at an isolated singular point P in R, and let C_1 be a circle in R with radius r_1 and with P as center. In previous papers* the writer has shown that in this neighborhood $f(x, y)$ can be put in the form

$$f(x, y) = c \log \frac{1}{r} + \Phi(x, y) + V(x, y), \tag{1}$$

where†

$$c = \frac{1}{2\pi} \int_{C_1} \frac{\partial f}{\partial n} ds,$$

r being the distance from (x, y) to P, $\Phi(x, y)$, unless it be identically zero, harmonic in the neighborhood of P and such that there exist modes of approach to P for which the sum $c \log (1/r) + \Phi$ tends toward plus infinity and also toward minus infinity; and V is harmonic everywhere in the neighborhood of P including P. Also on C_1, $\Phi \equiv 0$. It is to be noticed that the constant c may be zero so that Φ has the same properties ascribed to the sum $c \log (1/r) + \Phi$.

If a system of polar coordinates (r, θ) be chosen with P as pole, Φ may be expanded, for $r \leqq r_1$, in the form‡

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* G. E. Raynor, *Isolated singular points of harmonic functions*, this Bulletin, vol. 32 (1926), p. 543, and *Integro-differential equations of the Bôcher type*, this Bulletin, vol. 32, p. 654.

† Here, as in all that follows, the normal derivatives are to be taken in the direction of the inner normal.

‡ G. E. Raynor, *Note on the expansion of harmonic functions in the neighborhood of isolated singular points*, Annals of Mathematics, vol. 31 (1930), p. 40. We shall refer to this as paper (A).