A SET OF CYCLICLY RELATED FUNCTIONAL EQUATIONS*

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An examination of the first-order differential equation $y' = p(x)y$ and the second-order system $y' = p(x)y + q(x)z$, $z' = p(x)z + q(x)y$, where $p(x)$ and $q(x)$ are integrable functions of the real variable x, shows that their general solutions may be expressed in exponential form. Conversely, the solutions of the first-order equation may be used to define the exponential function and the solutions of the second-order system may be used to define the circular and hyperbolic functions as well as to give the relations that exist between these two sets of functions. These facts lead one to consider the general system

$$(1) \qquad y_i' = \sum_{k=1}^{n} A_k(x) y_{i+m+hk}, \quad (i = 1, \cdots, n;\ y_{j+n} \equiv y_j),$$

where n is a positive integer, m and h are integers or zero, and where the coefficients $A_k(x)$ are L-integrable functions of x on an interval of definition X. More generally, one is led to consider the functional system

$$(2) \qquad L(y_i) = \sum_{k=1}^{n} A_k y_{i+m+hk}, \quad (i = 1, \cdots, n;\ y_{j+n} \equiv y_j),$$

where the functional operator L has the property $L(ay + bz) \equiv aL(y) + bL(z)$ for any constants a and b, and where the coefficients A_k are functions of a finite or countably infinite set of variables $(x_1, x_2, \cdots)$ in a domain D of these variables. It is to be noted that the operator L may combine partial differential operators of various orders, simple and multiple integrals, and many other operators, so that system (1) occurs as a very special case of system (2). The present paper considers system (2). As a special case of the results of the paper, the general solution of system (1) is obtained and its exponential character exhibited.

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