DISCONTINUOUS SOLUTIONS IN THE CALCULUS OF VARIATIONS*

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1. *Introduction.* By a discontinuous solution in a calculus of variations problem is meant an extremizing arc having one or more corners, that is, the *derivatives* of the functions defining the arc have one or more ordinary discontinuities. Such discontinuities may occur when the extremizing curve is required to have a point in common with the boundary of the region where admissible curves must lie, or when the integrand function is discontinuous. These and other cases will be discussed later. But I wish to take up first the case when the function to be minimized (or maximized) is a simple integral of the form

$$J = \int_{t_1}^{t_2} F(x, y, x', y')dt,$$

where the integrand F has the usual continuity properties, and the minimizing curve is wholly interior to the region of admissible curves. When the problem is not regular, corners are very likely to occur on the minimizing arc, even when the two end-points can be joined by an extremal without corners. This depends on the behavior of the Weierstrass $\mathcal{E}$-function, as will be seen later. Examples are $F=(a\,x'^2+2\,b\,x'y'+c\,y'^2)^{1/2}$, where the quadratic form is positive definite and the coefficients a, b, c are properly chosen functions of x, y, x', y'; and, for a non-parametric problem, $f=y'^2(y'^2-\phi(x, y))$, where ϕ is positive.

2. *The Minimizing Curve is Interior to the Region of Admissible Curves.* The first additional necessary conditions on a minimizing arc with corners are the Weierstrass-Erdmann corner conditions, which state that $F_{x'}$ and $F_{y'}$ must be continuous along that arc. These conditions were given by Weierstrass in 1865.† For the non-parametric problem, the two functions

* An address presented at the invitation at the program committee at the meeting of the Society in Ann Arbor, November 29, 1929.

† (51), vol. 7, pp. 109, 250. The numbers in parentheses in foot notes refer throughout to the bibliographical list at the end.