A NOTE ON AN IMPORTANT THEOREM ON NORMAL DIVISION ALGEBRAS

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Dickson has shown that every normal division algebra A of order n^2 over a non-modular field F has rank n, in fact that its rank function $R(\omega; \xi_1, \cdots, \xi_m)$, $m = n^2$, is the characteristic equation of the general element $x = \sum \xi_i u_i$ of A in the expression of A as an algebra of n-rowed square matrices with scalar elements, an equation irreducible in $F(\xi_1, \cdots, \xi_m)$. He has also stated* a theorem upon which a great portion of the research in the theory of division algebras is based.

THEOREM. *Every normal division algebra A of order n^2 over a non-modular field F contains a quantity q whose minimum equation with respect to F has degree n.*

The proof given by Professor Dickson of the above theorem was an immediate application of *Hilbert's irreducibility theorem*, a theorem which has been shown true only for algebraic fields $R(x)$ over R, the field of all rational numbers. It is easy to give examples demonstrating that the property of Hilbert's theorem is not true for the fields of all complex numbers and all real numbers. A less obvious example is the case where F is the field of *all* numbers obtained in a finite number of steps from all rational numbers by the operations of addition, subtraction, multiplication, division except by zero, and extraction of *square* roots. For this case the equation $f(x, t) \equiv x^2 - t = 0$ is evidently irreducible in $F(t)$ when t is an indeterminate. But F is a field containing the square root of any number of F so that the above equation is reducible in F for every value of t in F and the property of Hilbert's theorem does not hold for F.

We shall give a simple algebraic proof of our normal division algebra theorem. The rank function of A is the characteristic equation of x in its expression as an n-rowed square matrix. By a well known theorem on matrices† the characteristic

* *Algebren und ihre Zahlentheorie*, pp. 263–4.

† Ibid., p. 20.