THEOREMS ON INVERTED AND ROTATED CONGRUENCES*

BY MALCOLM FOSTER

1. *Introduction.* Let l be any line of a rectilinear congruence and M that point on the unit sphere S at which the normal is parallel to l. We refer the sphere to any isothermal system and take the linear element in the form

$$ds^2 = e^{2\lambda}(du^2 + dv^2). \tag{1}$$

At M we consider the moving trihedral of S whose x-axis is chosen tangent to the curves $v=\text{const.}$, and let (a, b) be the coördinates of the point in which l meets the xy-plane. The conditions that a congruence l be of a particular type are conditions upon the functions a and b.†

For the sphere, $F=D'=0$, and hence

$$\xi_1 = \eta = p = q_1 = 0. \tag{2}$$

Also $E=\mathcal{E}$, $G=\mathcal{G}$, and $\rho_1=\rho_2=-1$;‡ consequently $D=-E$, $D''=-G=-E$. Hence we readily find§

$$\xi = \eta_1 = q = -p_1 = e^{\lambda},\ r = -\frac{\partial\lambda}{\partial v},\quad r_1 = \frac{\partial\lambda}{\partial u}. \tag{3}$$

When the functions in (3) are substituted in the six fundamental relations which are the equivalent of the Gauss and Codazzi equations,‖ all except the equation

$$\frac{\partial^2\lambda}{\partial u^2} + \frac{\partial^2\lambda}{\partial v^2} = -e^{2\lambda}¶ \tag{4}$$

* Presented to the Society, April 18, 1930.

† Malcolm Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), pp. 159–180.

‡ The positive direction of the normal is chosen outward.

§ Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 174.

‖ Eisenhart, p. 168 and p. 170.

¶ Foster, loc. cit. See p. 160 for the solution of this equation.