THE FIRST VARIATION OF A FUNCTIONAL*

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1. *Introduction.* The purpose of the present note is to present simple hypotheses which are sufficient to yield the two fundamental forms for the variation of a functional, expressed by a Stieltjes and a Lebesgue integral respectively.† The functional $F[f(\overset{b}{\underset{a}{x}})]$ is supposed to be defined for continuous functions $f(x)$ within a region R bounded by the continuous functions $\Phi_1(x)$, $\Phi_2(x)$, where $\Phi_1(x) < \Phi_2(x)$, and by the ordinates $x=a$, $x=b$:

$$\Phi_1(x) < f(x) < \Phi_2(x), \qquad a \leqq x \leqq b.$$

The following hypotheses are to be considered:

(I) There is an M_1 such that

$$|F[f_1] - F[f_2]| \leqq M_1 \max|f_1 - f_2|, \qquad (f_1, f_2 \text{ in } R).$$

(II) The first variation

$$D[f_1,\phi] = \lim_{\epsilon=0} \frac{F[f_1+\epsilon\phi] - F[f_1]}{\epsilon}$$

exists, and the limit so defined exists uniformly for all $f_1(x)$ in R, where $\phi(x)$ is an arbitrary given continuous function, $a \leqq x \leqq b$.

(III) There is an M such that

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† The reader may consult the following references for various types of sufficient conditions:

V. Volterra, *Les Fonctions de Lignes*, Paris, 1911.

F. Riesz, *Concernant les opérations fonctionnelles linéaires*, Annales de l'Ecole Normale Supérieure, vol. 31, p. 10. See also F. Riesz, *Sur les opérations lineaires* (troisième note), Annales de l'Ecole Normale Supérieure, vol. 8 (1907), p. 439.

M. Fréchet, *Sur la notion de différentielle de fonction de ligne*, Transactions of this Society, vol. 15 (1914), p. 139.

G. C. Evans, *Note on the derivative and the variation of a function depending on all the values of another function*, this Bulletin, vol. 21 (1915), p. 387.

P. J. Daniell, *The derivative of a functional*, this Bulletin, vol. 25 (1919), p. 414.