SOME PROPERTIES OF SPHERICAL HARMONICS*

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A Newtonian potential $V(x, y, z)$ can often be derived from a four-dimensional potential $W(x, y, z, w)$ by forming the definite integral

$$V = \frac{1}{\pi}\int_{-\infty}^{\infty} W dw.$$

Thus, if W is the reciprocal of $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(w-w_0)^2$, where x_0, y_0, z_0, w_0 are real constants, the Newtonian potential V is the inverse of the distance between the points (x, y, z) and (x_0, y_0, z_0). There is thus a simple correspondence between the *charges* giving rise to the two potentials, the point charge in the three-dimensional space, S_3, being simply the projection of the corresponding point charge in the four-dimensional space S_4. With suitable restrictions this method of projection may be applied to surface distributions of charge in S_4 and we shall consider in particular the case of a continuous distribution over the spherical surface

$$x^2 + y^2 + w^2 = a^2, \quad z = 0,$$

when the surface density depends only on x^2+y^2. In this case

$$W = \int_0^{\pi}\int_0^{2\pi} (a^2/R^2) f(\cos\theta) \sin\theta d\theta d\phi ,$$

where $f(\cos\theta)$ is the function giving the law of density and

$$R^2 = (w - a\cos\theta)^2 + z^2 + (x - a\sin\theta\cos\phi)^2 + (y - a\sin\theta\sin\phi)^2.$$

To reduce the integral to a simpler form we write

$$\cos\theta = \zeta, \qquad x^2 + y^2 = \rho^2, \qquad \rho^2 + w^2 = \tau^2;$$

then

$$W = (\pi a/\tau) U(X, Y, Z),$$

where

* Presented to the Society, June 20, 1929.