MATRICES WHOSE CHARACTERISTIC EQUATIONS ARE CYCLIC*

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One of Sylvester's theorems† on matrices states that if the characteristic equation

$$|M - \lambda I| = f(\lambda) = 0 \tag{1}$$

of a square matrix M has the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the characteristic equation

$$|\phi(M) - \rho I| = g(\rho) = 0 \tag{2}$$

of any integral function of M, namely, $\phi(M)$, has the roots $\rho_i = \phi(\lambda_i)$, $i = 1, 2, \cdots, n$. In this note an isomorphism is shown to exist between the algebraic and matric roots of (1) when this equation is cyclic. Certain consequences of this isomorphism are given. Since

$$\begin{pmatrix} -a & -b \cdots & -k & -l \\ 1 & 0 \cdots & 0 & 0 \\ 0 & 1 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \cdots & 1 & 0 \end{pmatrix}$$

is the matrix which has $\lambda^n + a\lambda^{n-1} + b\lambda^{n-2} + \cdots + k\lambda + l = 0$ as its characteristic equation,‡ Sylvester's theorem furnishes a method of effecting the Tschirnhaus transformation $\rho = \phi(\lambda)$ on any equation. When (1) is cyclic an especially interesting type of Tschirnhaus transformation is possible.

Suppose (1) to be a cyclic equation of degree n with the relations $\lambda_{i+1} = \phi(\lambda_i)$, $i = 1, 2, \cdots, n$ and $\lambda_{n+j} = \lambda_j$ connecting its roots. For simplicity of notation let ϕ_i be the ith iterated function of ϕ so that $\lambda_2 = \phi(\lambda_1) = \phi_1(\lambda_1)$, $\lambda_3 = \phi(\lambda_2) = \phi(\phi(\lambda_1)) = \phi_2(\lambda_1)$, and in general $\lambda_{i+1} = \phi_i(\lambda_1)$. By Sylvester's theorem the roots of (2) then are

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† Sylvester, *Mathematical Papers*, vol. 4, p. 133; Frobenius, Journal für Mathematik, vol. 84, p. 11; Dickson, *Algebren und ihre Zahlentheorie*, p. 18.

‡ Wedderburn, Annals of Mathematics, vol. 27 (1926), p. 247.