HAUSDORFF'S THEOREM CONCERNING HERMITIAN FORMS

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In 1919, Hausdorff proved an interesting theorem to the effect that the complex values assumed by the Hermitian form

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\bar{x}_\alpha x_\beta,$$

when the complex numbers $x_1, \cdots, x_n$ take on all values compatible with the relation

$$\sum_{\alpha=1}^{n} \bar{x}_\alpha x_\alpha = 1,$$

constitute a convex point set in the complex plane.* In his demonstration Hausdorff employs the transformation of Hermitian symmetric forms of this type to principal axes. It is obvious, therefore, that the extension of this method to more general instances is impossible, because the principal axis transformation may no longer be available, a situation which arises, for instance, in the important case of unbounded forms in infinitely many variables. On this account, a proof along less special lines is of considerable interest. It is gratifying to ascertain that an entirely elementary, explicit, and general demonstration of Hausdorff's theorem can be devised by making it depend upon the theorem for binary forms. The proof we shall give involves nothing more difficult than manipulations of complex numbers and the solution of quadratic equations.

We first indicate what we shall mean, abstractly, by an Hermitian bilinear form. Let $\mathfrak{H}$ be a class of elements in which the operations $+$ (vector addition) and $a\cdot$ (scalar multiplication by an arbitrary complex constant a) are significant and possess their customary algebraic properties; let $\mathfrak{H}$ be closed under these operations. Such a class may be called a complex vector space, its elements complex vectors. A complex-valued

* Mathematische Zeitschrift, vol. 3 (1919), pp. 314–316.