ON POLYNOMIAL SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

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1. *Introduction.* Certain well known polynomials have a number of common properties. They arise as coefficients of t^n in the expansion of a generating function; they may be obtained by means of orthogonalization of a set of functions $x^n g(x)$ when the function $\rho(x)=g^2(x)$ and the interval are properly chosen; they may be regarded as polynomials which become orthogonal when multiplied by a proper factor $g(x)$; they satisfy a certain type of difference equation; they satisfy a certain type of differential equation. The results of this paper are based on the differential equation. Some of them are general statements of results already known for various classes of polynomials; others are believed to be new.

2. *Polynomial Solutions.* Consider the differential equation

$$P(x)y_n'' + Q(x)y_n' + \lambda_n R(x)y_n = 0, \tag{1}$$

where the coefficients P, Q, R are assumed to be real polynomials in the real variable x none of which vanish identically and λ_n is a polynomial in n. We assume that y_n is a polynomial of the form†

$$y_n = x^n + C_1^{(n)}x^{n-1} + C_2^{(n)}x^{n-2} + \cdots + C_n^{(n)}, \quad (n = 0,1,2,\cdots), \tag{2}$$

where the coefficient of the highest power of x is unity. On substituting $y_0=1$, $y_1=x+a_1$, $y_2=x^2+ax+b$, we find that P and Q must contain the factor R. Dividing out this factor and putting $\lambda_1=1$, as may be done without loss of generality, we find that the differential equation must have the form

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† When reference is made to various standard polynomials it will be assumed that they have been put into this form by use of a suitable factor.