ON A FUNCTION CONNECTED WITH $\phi(n)$

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Let $\phi(n)$ be the number of numbers not greater than and prime to n. Further, let

$$\begin{aligned} \phi_1(n) &= \phi(n), \\ \phi_2(n) &= \phi\{\phi_1(n)\}, \\ \phi_3(n) &= \phi\{\phi_2(n)\}, \\ & \cdots\cdots\cdots \\ \phi_{r+1}(n) &= \phi\{\phi_r(n)\}. \end{aligned}$$

It is obvious that at one stage

$$\phi_r(1) = 1.$$

Hence, with an integer n, there is associated another $R(n) = r$, such that r is the least integer for which

$$\phi_r(n) = 1.$$

This short note is about this function $R(n)$.*

THEOREM I.

$$R(n) \leqq \left[\frac{\log n}{\log 2}\right] + 1.$$

PROOF. When $n > 2$, $\phi(n)$ is always even. If x and y are even, and y contains at least one odd prime factor and x does not, then

$$\frac{\phi(x)}{x} > \frac{\phi(y)}{y}.$$

Hence, when n is even, $R(n)$ is maximum only if n is a power of 2.

If n is a power of 2,

* This problem was suggested by R. Vaidyanathaswami.