NON-ISOLATED CRITICAL POINTS OF FUNCTIONS*

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The isolated critical points of real functions of n independent real variables have been treated in an elegant manner by Marston Morse.‡ This treatment obtains definite relations between the numbers of critical points of $n+1$ types that appear in a bounded portion of the space of the independent variables. Morse requires his functions to have continuous third partial derivatives in the neighborhoods of the critical points and imposes conditions that are sufficient to insure the existence of at most a finite number of such points in the domain under consideration. In the present note§ we consider functions that have continuous first partial derivatives and may have an infinite number of critical points, or even continua of such points, in the given domain. As a special case of our results we obtain the *minimax principle* of Birkhoff in the modified form given by Bieberbach.‖

Let S denote the space of the n real variables $x_1, x_2, \cdots, x_n$, and let $(x)=(x_1, x_2, \cdots, x_n)$ denote a point in this space. Let R be a bounded,¶ open, and connected point set in S and let C, the boundary of R, be connected. Let the real function $f(x_1, \cdots, x_n)$ be single-valued and continuous

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‖ *Differentialgleichungen*, Berlin, Springer, 1927, p. 140.

¶ The terms of classical point set theory are used in their usual sense. The distance between two points is given by a generalization of the ordinary distance formula in the plane. If K denotes a point set, then K' is used to denote K together with all of its limit points.