THE APPROXIMATION OF HARMONIC FUNCTIONS BY HARMONIC POLYNOMIALS AND BY HARMONIC RATIONAL FUNCTIONS*

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1. *Introduction.* The following theorem of Weierstrass is classical.

Let the function $f(\theta)$ be continuous for all values of the argument and periodic with period 2π. Then if an arbitrary positive ϵ be given, there exists a trigonometric polynomial which differs from $f(\theta)$ at most by ϵ; that is, the inequality

$$\left| f(\theta) - \sum_{k=0}^{n}(a_k \cos k\theta + b_k \sin k\theta) \right| \leqq \epsilon \tag{1}$$

holds for all values of θ.

An equivalent statement of the conclusion of this theorem is that *$f(\theta)$ can be expanded in a series of the form*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n}(a_{nk} \cos k\theta + b_{nk} \sin k\theta),$$

which converges uniformly for all values of θ. This is of course a general fact, that if a given function can be uniformly approximated as closely as desired by a linear combination of other functions, then that function can be expanded in a uniformly convergent series of which each term is a linear combination of those other functions, and conversely. This fact is easy to prove and will be frequently used in the sequel.

If in the (x, y)-plane we introduce polar coordinates (r, θ) and consider the function $f(\theta)$ defined on the unit circle C, Weierstrass's theorem refers to the approximation of a function $f(\theta)$ continuous on C by trigonometric polynomials, or what is the same thing, by polynomials of the form

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