A PARTIAL ISOMORPHISM BETWEEN THE FUNCTIONS OF LUCAS AND WEIERSTRASS*

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1. *Introduction.* In a former paper† it was pointed out that certain identities in the theory of multiplication, real or complex, of elliptic functions, are of precisely the same form as others between Lucas' U_n, V_n. The latter were defined only for integer values of n by Lucas as follows,

$$U_n \equiv \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n \equiv \alpha^n + \beta^n,$$

where α, β are real or complex numbers such that $\alpha+\beta=p$, $\alpha\beta=q$, where p, q are rational numbers. Hence U_n, V_n are rational numbers and are solutions of

$$W_{n+2} = pW_{n+1} - qW_n,$$

with the initial conditions

$$(U_0, U_1) = (0, 1), \quad (V_0, V_1) = (2, p).$$

It follows, as shown by Lucas,‡ that U_n, V_n are simply expressible as circular functions of a real or imaginary argument, according as α/β is imaginary or real. Thus every formula in elliptic functions will contain as a degenerate case one concerning the U_n, V_n functions, but this is not the isomorphism sought.

We first generalize Lucas' definition in one respect and specialize it in another, replacing n by the complex variable z and restricting q to be 1. We shall write the functions thus defined as

$$U(z) \equiv \frac{\alpha^z - \alpha^{-z}}{\alpha - \alpha^{-1}}, \quad V(z) \equiv \alpha^z + \alpha^{-z}, \tag{1}$$

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† This Bulletin, vol. 29 (1923), pp. 401–406.

‡ American Journal of Mathematics, vol. 1, (1878), p. 189.