THE FORMS $ax^2+by^2+cz^2$ WHICH REPRESENT ALL INTEGERS

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THEOREM. *$f=ax^2+by^2+cz^2$ represents all integers, positive, negative, or zero, if and only if*: I. *a, b, c are not all of like sign and no one is zero*; II. *no two of a, b, c have a common odd prime factor*; III. *either a, b, c are all odd, or two are odd and one is double an odd*; IV. *$-bc$, $-ac$, $-ab$ are quadratic residues of a, b, c, respectively.*

We shall first prove that I-IV are necessary conditions. Let therefore f represent all integers. It is well known that I follows readily.

If a and b are divisible by the odd prime p, f represents only $1+\frac{1}{2}(p-1)$ incongruent residues cz^2 modulo p. This proves II.

Next, *no one of a, b, c is divisible by* 8. Let $a\equiv 0 \pmod 8$. Every square is $\equiv 0$, 1, or 4 (mod 8). First, let $b=2B$. Since f represents odd integers, c is odd. Since $by^2\equiv 0$ or $2B$ (mod 8) and $cz^2\equiv 0$, c, or $4c$, f has at most six residues modulo 8. If m is a missing residue, f represents no $m+pn$. Second let b and c be odd. Then $4b\equiv 4c\equiv 4 \pmod 8$. Thus the residues of f modulo 8 are obtained by adding each of 0, 4, b to each of 0, 4, c; we get only seven residues 0, 4, b, c, $4+b$, $4+c$, $b+c$.

No one of a, b, c is divisible by 4. Let a be divisible by 4. Since a is not divisible by 8, $a\equiv 4 \pmod 8$. Evidently $f\equiv 0$, b, c, or $b+c$ (mod 4). No two of these are congruent modulo 4. If $b\equiv \pm 1 \pmod 4$, they are 0, ± 1, c, $c\pm 1$. Evidently c is not congruent to 0, ± 1, or ∓ 1. Hence $c\equiv 2 \pmod 4$. Since $b\not\equiv 0$, this proves that one of b and c is $\equiv 2 \pmod 4$. By symmetry, we may take $b\equiv 2 \pmod 4$. If $b\equiv 6 \pmod 8$, we apply our discussion to $-f$ instead of