ON LEVI-CIVITA'S THEORY OF PARALLELISM

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1. *Parallel Displacements.* The use of two-parametric differential invariants* (grad, div etc.) for a surface not only leads very neatly and quickly to the condition of parallelism and the known theorems, but also puts in evidence other results hitherto unnoticed. In particular, some interesting properties of Tchebychef nets are discovered.

Consider a curve C drawn on a surface S in ordinary space. Let $\boldsymbol{R}$ be a vector of constant length (which may be taken as unity), and a point-function along C, being everywhere tangential to the surface. Then $\boldsymbol{R}$ is said to undergo a parallel displacement along C, or to remain parallel to itself for displacement along this curve, provided the derivative of $\boldsymbol{R}$ along C is everywhere normal to the surface.† We lose no generality by regarding C as a member of a family T of curves, with unit tangent $\boldsymbol{t}$ and arc-length s, and $\boldsymbol{R}$ as the unit tangent to a second family cutting the former at a variable angle θ, so that θ is the angle of rotation from $\boldsymbol{t}$ to $\boldsymbol{R}$ in the positive sense about the unit normal $\boldsymbol{n}$. If then $\boldsymbol{m}$ is the unit surface vector $\boldsymbol{n}\times\boldsymbol{t}$, perpendicular to $\boldsymbol{t}$, we may write

$$\boldsymbol{R} = \boldsymbol{t}\cos\theta + \boldsymbol{m}\sin\theta.$$

Let a prime denote differentiation with respect to s, that is to say, in the direction of $\boldsymbol{t}$. Then in order that $\boldsymbol{R}$ may be normal to S we must have

$$0 = \boldsymbol{t}\cdot\boldsymbol{R}' = \sin\theta[\boldsymbol{t}'\boldsymbol{m}\cdot - \theta'],$$

and

$$0 = \boldsymbol{m}\cdot\boldsymbol{R}' = \cos\theta[\boldsymbol{m}\cdot\boldsymbol{t}' + \theta'],$$

* All the differential invariants of this article are two-parametric invariants. See the writer's *Differential Geometry*, Chap. 12, Cambridge University Press, or Quarterly Journal of Mathematics, vol. 50, pp. 230–269.

† See Levi-Civita, *The Absolute Differential Calculus*, pp. 101–119.