ON THE EXISTENCE OF LINEAR ALGEBRAS IN BOOLEAN ALGEBRAS*

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According to the definition to be found in Dickson's *Algebras and their Arithmetics*, pages 9–11, the following properties are characteristic of a linear algebra.

(1) *The elements of the algebra form an abelian group with respect to addition.*

(2) *Multiplication is distributive with respect to addition.*

(3) *The algebra has a finite basis; that is, a finite number of elements can be found such that every element of the algebra can be expressed as a linear combination of these basal units, with coefficients taken from the field over which the algebra is defined.*

With this definition in mind we wish to determine for what pairs of Boolean operations considered as the addition and multiplication operations the elements of a Boolean algebra will constitute a linear algebra. We are limited in our choice of an addition operation by the first property above to operations of the form $axy+a'xy'+a'x'y+ax'y'$, which Bernstein‡ has shown to be the only Boolean abelian group operations. To find suitable multiplication operations $pxy+qxy'+rx'y+sx'y'$, we seek those which are distributive with respect to the above. The condition that the first distributive law hold is found in Schröder's *Algebra der Logik* (vol. 2, p. 503) to be

$$a'(pq+rs)+(ad+a'd'+bc+b'c')(p'q+r's)+d(p'q'+r's')=0,$$

where a, b, c, d, and p, q, r, s are the discriminants of the addition and multiplication operations respectively. Here $a=d$, and $b=c=a'$; hence we get $p'q=r's=a'q=a's=ap'$

* Presented to the Society, January 1, 1926.

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‡ Transactions of this Society, vol. 26, p. 174.