A NEW NORMAL FORM FOR QUARTIC EQUATIONS*

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1. *Introduction.* There are a number of well known normal forms to which the general quartic equation may be reduced by means of a Tschirnhaus transformation, without requiring the solution of any equation of higher than the third degree. The reduction to $y^4+c_3y+c_4=0$ or to $y^4+c_2y^2+c_4=0$ requires no comment. Bring first showed how to obtain the form $y^4+c_1y^3+c_4=0$, by applying a simple reciprocal transformation (which is reducible to a Tschirnhaus transformation) to the first form above.† The reduction to the binomial form $y^4+c_4=0$ by an ordinary third degree transformation leads to a sixth degree equation, but Lagrange was able to show (by an a priori proof) that this sextic will factor into three quadratics whose coefficients are themselves roots of cubic equations.‡ The required factorization would, however, be difficult actually to obtain; the necessary transformation may be obtained more conveniently in another way.§

I wish in this paper to discuss a new normal form, $y^4+c_2y^2=0$. I first give an a priori proof, based on that of Lagrange for the transformation to binomial form, that the reduction does not involve the solution of any equation of higher than the third degree. Then, using a theorem of Hermite and certain results of Cayley, I consider the setting up of the necessary transformation.

2. *The a Priori Proof.* If we apply the transformation

$$y = x^3 + k_2x^2 + k_3x + k_4 \tag{1}$$

to the general quartic equation

* Presented to the Society, April 7, 1928.

† Bring's work was published in 1786. It is reproduced in Grunert's Archiv, vol. 41 (1864), pp. 105–112.

‡ See his *Oeuvres*, vol. III, Paris, 1869, pp. 284–295. His result was originally published in 1770.

§ See this Bulletin, vol. 33 (1927), No. 6.