NOTE ON A CONVERGENCE PROOF

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Some years ago I published a particularly simple proof of the convergence of the Fejér mean of the Fourier series for an arbitrary continuous function.* I did not notice until some time later that the same proof had already been given by Haar† in his thesis. The present note constitutes a renewed attempt to contribute something to the theory of the method in question, by applying it to a problem which is not treated by Haar, in the passage cited at any rate. The substance of the note consists in the proof of the following theorem:‡

Let $f(x)$ be an arbitrary continuous function of period 2π. With each positive integral value of n, let an integer m_n be associated, subject merely to the condition that $m_n \geqq n$, and let

$$\tau_n(x) = \frac{1}{nm_n} \sum_{i=1}^{m_n} f(t_i) \frac{\sin^2 \frac{1}{2}n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)}, \tag{1}$$

where $t_i = 2i\pi/m_n$. Then $\tau_n(x)$ converges uniformly toward $f(x)$ as n becomes infinite.

The reasoning is given in full, so that it can be understood

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* *Note on a method of proof in the theory of Fourier's series*, this Bulletin, vol. 27 (1920–21), pp. 108–110.

† A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Dissertation, Göttingen, 1909; p. 29; reprinted in Mathematische Annalen, vol. 69 (1910), pp. 331–371; pp. 353–354.

‡ For the case $m_n = n$, see D. Jackson, *A formula of trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 371–375; S. Bernstein, *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus, vol. 158 (1914), pp. 1661–1663; L. Fejér, *Über Interpolation*, Göttinger Nachrichten (1916), pp. 66–91; pp. 87–91. For a corresponding generalization of the ordinary formula of trigonometric interpolation, see D. Jackson, *Some notes on trigonometric interpolation*, American Mathematical Monthly, vol. 34 (1927), pp. 401–405. For the underlying idea of the present treatment, see also Hahn, *Über das Interpolationsproblem*, Mathematische Zeitschrift, vol. 1 (1918), pp. 115–142.