CAUCHY'S CYCLOTOMIC FUNCTION AND FUNCTIONAL POWERS*

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1. *Introduction.* Let $\phi(n)$ as usual denote the totient of n. Cauchy's function is the polynomial $F_n(x)$ of degree $\phi(n)$, whose zeros are the primitive nth roots of unity. If $a, b, c, e, \cdots$ are the distinct prime divisors (>1) of n,

$$F_n(x) = N_n(x)/D_n(x), \tag{1}$$

where

$$N_n(x) \equiv (x^n - 1)\prod(x^{n/ab} - 1)\prod(x^{n/abce} - 1)\cdots,$$
$$D_n(x) \equiv \prod(x^{n/a} - 1)\prod(x^{n/abc} - 1)\cdots.$$

An anonymous writer† in L'Intermédiaire des Mathématiciens (vol. 24 (1917), pp. 5–6), stated that $\prod F_d(1) = n$, where d ranges over all divisors of n, also that the value of $F_n(-1)$ is 2 if n is a power of 2, $F_p(1)$ if n is double an odd prime p, but is 1 in all remaining cases. The second part seems to be incorrect. For example, $F_2(x) = x+1$, $F_{18}(x) = x^6 - x^3 + 1$, $F_{50}(x) = x^{20} - x^{15} + x^{10} - x^5 + 1$, and hence for $n = 2, 18, 50$ the respective values of $F_n(-1)$ are 0, 3, 5, not 2, 1, 1 as demanded by the assertion. To obtain in §3 the correct values of $F_n(-1)$ it is convenient to prove in §2 the first result stated, which in turn depends upon a theorem of Trudi, which we shall also prove in §2 as several details of the proof are useful. Finally, in §§4, 5 we connect $F_n(x)$ in a new way with other numerical functions, to illustrate what are here called *functional powers*.

We shall need $G_n(x)$ defined by

$$G_n(x) = M_n(x)/R_n(x), \tag{2}$$

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† Quoted from the *Report on Algebraic Numbers*, Bulletin of the National Research Council, vol. 5, Part 3, No. 28 (1923), p. 32, where it is also stated that no reply is given in L'Intermédiaire for 1917–19. References to other writers cited here will be found on pp. 31, 32 of the *Report*.