AN ELEMENTARY PROOF BY MATHEMATICAL INDUCTION OF THE EQUIVALENCE OF THE CESÀRO AND HÖLDER SUM FORMULAS*

BY TOMLINSON FORT

For brevity, notation and terminology in this note are generally not explained. It is believed that they will be clear in all instances to any probable reader.

THEOREM. *The Hölder and Cesàro methods of summation are equivalent.*

PROOF. Let $C_n^{(2)}$ represent the Cesàro sum of order r to n terms:

$$(1)\qquad C_n^{(r)} = s_0\frac{r}{r+n} + \cdots + s_k\frac{r(n-k+1)\cdots n}{(r+n-k)\cdots(r+n)} + \cdots + s_n\frac{r(n!)}{r\cdots(r+n)}.$$

We readily verify that $C_n^{(r)}$ satisfies the equation

$$(2)\qquad (n+r+1)\,C_n^{(r+1)} - nC_{(n-1)}^{(r+1)} = (r+1)C_n^{(r)},$$

which may be written in the form

$$\Delta\{nC_{(n-1)}^{(r+1)}\} + r\,C_n^{(r+1)} = (r+1)C_n^{(r)},$$

or

$$(3)\qquad (n+1)C_n^{(r+1)} + r\sum_{n=0}^{n} C_n^{(r+1)} = (r+1)\sum_{n=0}^{n} C_n^{(r)}.$$

By solving (2) we get the following, as is easily verified:

$$C_n^{(r+1)} = \frac{n!}{(r+2)\cdots(r+n+1)}\sum_{n=0}^{n}\frac{(r+1)\cdots(r+n)}{n!}C_n^{(r)}$$

$$(4)\qquad = \frac{(r+1)!}{(n+1)\cdots(n+r+1)}\sum_{n=0}^{n}\frac{(n+1)\cdots(n+r)}{r!}C_n^{(r)}.$$

*Presented to the Society, December 29, 1926.