

ON A GENERALIZATION OF THE
SECULAR EQUATION*

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1 *Introduction.* The equation we wish to consider is

$$(1) \quad H_n(x) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

Here $a_{ij} = a_{ji}$, when $i \neq j$; while

$$\begin{aligned} a_{ii} &= \alpha_{ii} - x, & \text{for } i = 1, 2, \dots, r, \\ &= \alpha_{ii} + x, & \text{for } i = r + 1, r + 2, \dots, n. \end{aligned}$$

The a_{ij} and α_{ii} are real, and $H(0) \neq 0$. If $r = n$, (1) is the secular equation which it will be convenient to denote by $L_n(x) = 0$. When $r = n - 1$ the equation (1) plays a fundamental role in classifying quadric surfaces in n -way hyperbolic space. Let us set $n - r = s$ and call $\sigma = |r - s|$ the *signature* of (1). We have then the

THEOREM I. *The number of real roots of $H_n(x) = 0$, counting their multiplicity, is not less than its signature.*

This is a corollary of a theorem to which F. Klein calls especial attention (*Mathematische Annalen*, vol. 23 (1884), p. 562). The proof there given rests on the theory of elementary divisors.† We give here a very simple proof which is a modification of H. Weber's proof that the roots of the secular equation $L_n(x) = 0$ are all real.‡ Weber's proof as we shall see, is complicated by his belief that it is necessary to show that

$$L_n'(x) = - \sum_i \frac{\partial L_n}{\partial a_{ii}}, \quad (i = 1, 2, \dots, n).$$

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† See T. J. Bromwich, *Quadratic Forms*, Cambridge Tracts, No. 3 (1906), p. 69.

‡ H. Weber, *Algebra*, vol. 1, 1898, pp. 307-310.