AN ASSEMBLAGE-THEORETIC PROOF OF THE EXISTENCE OF TRANSCENDENTALLY TRANSCENDENTAL FUNCTIONS*

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1. *Introduction.* The first example of a transcendentally transcendental function,† that is, of an analytic function not a solution of any algebraic differential equation, was given by Hölder,‡ who, in 1887, showed that the gamma function does not satisfy any such equation. Other investigations have followed Hölder's. In some of these, functional equations of different types are studied, to find which of their solutions satisfy algebraic differential equations. In others, e.g., in that of Hurwitz, on power series with rational coefficients§ and in that of Ostrowsky on Dirichlet series,|| certain analytic expressions are examined and conditions found under which they represent transcendentally transcendental functions.

In the present note, the existence of transcendentally transcendental functions is shown on an a priori basis, by a proof resembling that proof of the existence of transcendental numbers which is based on the countability of the algebraic numbers. The totality of algebraic differential equations has the same power as the totality of analytic functions, but there exists a countable set of the equations, namely, those with integral coefficients, whose solutions form the totality of the solutions of all algebraic differential equations. It follows that the solutions of the countable set of equations do not exhaust the analytic functions.

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† This term is due to E. H. Moore, Mathematische Annalen, vol. 48 (1896), p. 49.

‡ Mathematische Annalen, vol. 28 (1887), p. 1.

§ Annales de l'Ecole Normale Supérieure, vol. 6 (1889), p. 327.

|| Mathematische Zeitschrift, vol. 8 (1920), p. 241.