ON THE INTEGRATION IN FINITE TERMS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

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1. *Introduction.* Liouville, in 1840, investigated the equation

$$\frac{d^2w}{dz^2} = \chi(z)w, \tag{A}$$

with $\chi(z)$ an elementary function of z, to determine under what circumstances the solutions of the equation are elementary.†

By an "elementary function", Liouville understood, in this connection, any function of z obtained in a finite number of steps by performing algebraic operations, taking logarithms and exponentials, and performing integrations. An example of such a function would be

$$\int \left[\log \arcsin z + \int z^2 dz\right]^{1/2} dz + \tan \log_z [1 + (z)^{1/2}].$$

It is a consequence of Liouville's work that the solutions of Bessel's equation

$$z^2w'' + zw' + (z^2 - \nu^2)w = 0$$

are elementary functions only when 2ν is an odd integer.‡ But it would be improper to conclude from Liouville's results alone, as some authors seem to do, that Bessel's equation cannot be solved by the simpler formal methods of the theory of differential equations. For instance, the equation

$$\frac{du}{dz} = \frac{u}{u-1}$$

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† Journal de Mathématiques, vol. 5 (1840). See also Watson, *Theory of Bessel Functions*, Cambridge, 1922, p. 111. No acquaintance with Liouville's work is necessary for the understanding of the present paper.

‡ Watson, loc. cit.