ON THE CONVERGENCE OF TRIGONOMETRIC APPROXIMATIONS FOR A FUNCTION OF TWO VARIABLES*

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A discussion of the convergence of approximating functions for a given function of two variables $f(x, y)$, just as in the case of functions of one variable, can be based on two sets of theorems: (1) theorems on the existence of functions of closest approximation; (2) theorems on the representation of $f(x, y)$ by means of finite sums constructed in a specific way.

From the first group we shall make use of the following theorem:

THEOREM I. *Let $p_1(x,y)$, $p_2(x\ y)$, $\cdots$, $p_N(x,y)$ be N functions of x and y, continuous in the region R: $(a \leqq x \leqq b$, $c \leqq y \leqq d)$, and linearly independent in this region. Let*

$$\phi(x,y) = c_1 p_1(x,y) + c_2 p_2(x,y) + \cdots + c_N p_N(x,y)$$

be an arbitrary linear combination of the given functions with constant coefficients. Let $f(x, y)$ be a continuous function of x and y in R. Then there exists a choice of the coefficients c_k in $\phi(x, y)$ such that the integral

$$\int_a^b dx \int_c^d |f(x,y) - \phi(x,y)|^m dy, \qquad (m > 0),$$

has its minimum value. The function $\phi(x, y)$ so determined is unique for $m > 1$. It is called an approximating function for $f(x, y)$ corresponding to the exponent m.

This theorem can be proved by methods analogous to those used in proving the corresponding theorem for functions of a single variable.† In this paper we shall choose

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† Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128; *Note on an ambiguous case of approximation*, TRANSACTIONS, vol. 25 (1923), pp. 333–337.