A BOUNDARY VALUE PROBLEM IN THE CALCULUS OF VARIATIONS*

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1. *Introduction.* It is well known that the necessary condition of Jacobi for a problem with fixed end-points in the calculus of variations is closely related to a certain boundary value problem associated with Jacobi's differential equation.† If Jacobi's condition is satisfied then the smallest value λ_1 of the parameter λ of the boundary value problem, for which that problem has a solution, must satisfy the inequality $\lambda_1 \geqq 0$, and conversely. In the following pages it is proposed to deduce a similar relationship for problems of the calculus of variations with variable end-points. The corresponding boundary value problem has end conditions of a more general type than those which arise when the end-points of the original calculus of variations problem are fixed. The existence of the smallest value λ_1 is established by methods of the calculus of variations, in particular by means of a theorem analogous to a well known theorem of Osgood.‡ It seems likely that a complete theory of self-adjoint boundary value problems for ordinary differential equations, with end conditions of a very general type, can be deduced from theorems already well known in the calculus of variations. So far as I know this has never been done, though many significant relationships have of course been pointed out.§

2. *The Calculus of Variations Problem and its Second Variation.* Let C_1 and C_2 be two arcs with the parametric equations

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† See, for example, Lovitt, *Linear Integral Equations*, p. 207.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 2 (1901), p. 273.

§ See, for example, Richardson, MATHEMATISCHE ANNALEN, vol. 68 (1910), p. 279; Plancherel, BULLETIN DES SCIENCES MATHÉMATIQUES, vol. 47 (1923), p. 376.