NOTE ON A PAIR OF PROPERTIES WHICH CHARACTERIZE CONTINUOUS FUNCTIONS

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In a paper which appeared in this BULLETIN* D. C. Gillespie discusses the class of functions of a single variable which share with continuous functions the property of never passing from one value to another without taking every intermediate value. He proves that a function of this class will be continuous if the set of values which it assumes an infinite number of times does not completely fill any interval, but he points out that this condition is not a necessary one. This leads us to ask whether we can form a condition for the continuity of a function of this class which is both necessary and sufficient, but which does not imply the continuity of a function which is otherwise unrestricted. An anwer to this question is given by the following theorem.

Necessary and sufficient conditions for the continuity of a function $f(x)$ defined in a closed interval (a, b) are the following:

(A) If x_1 and x_2 are any two points of the interval, $f(x)$ takes each value between $f(x_1)$ and $f(x_2)$ in the interval (x_1, x_2).

(B) For every value of α, the set $E\ (f=\alpha)$† is closed.

That these conditions are necessary is well known,‡ and that (B) alone does not imply continuity is shown by the example of any monotone discontinuous function, not constant in any interval. It thus remains to prove that the conditions are sufficient.

Let ξ be any point in the interval and ϵ a given positive number. Denoting $f(\xi)$ by η, consider the sets $E(f=\eta+\epsilon)$

* Vol. 28 (1922), p. 245.

† This symbol denotes the set consisting of those points of the region in which the function is being considered at which the function is equal to α.

‡ See, e.g., W. H. Young, *The Fundamental Theorems of the Differential Calculus*, Cambridge Mathematical Tracts, No. 11, p. 7.