NOTE ON A FUNDAMENTAL LEMMA CONCERNING THE LIMIT OF A SUM*

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In a recent paper† the author presented a proof of a lemma which is of fundamental importance in the theory of the Riemann integral. It is the purpose of this note to replace the proof of Case I of the above paper by the present one which treats the case of unequal subintervals. The theorem by means of which this proof is completed is of interest in itself, and admits of certain generalizations, one of which will be stated in the sequel.

A *horizontal* function of index n, $h(x, n)$, is a function defined on an interval $I\equiv(a\leqq x\leqq b)$ in such a manner that it is possible to divide I into n subintervals $I(i, n)$, $(i=1, 2, \cdots, n)$, so that $h(x, n)\equiv h_{in}\equiv$constant, at all interior points of $I(i, n)$. In addition, we define $h(a, n)\equiv h_{1n}$ and $h(b, n)\equiv h_{nn}$. Let $x(i, n)$ be the $n-1$ distinct internal points of division which subdivide I and together with $a\equiv x(0, n)$, $b\equiv x(n, n)$ form the subdivisions, $I(i, n)$, whose lengths are $\Delta_{in}=x(i, n)-x(i-1, n)$. We shall define the value of $h(x, n)$ at the internal points of division $x(i, n)$ to be that value of the pair of numbers h_{in}, $h_{i+1,n}$, whose numerical value is the larger. Furthermore, let

$$f(n)=\sum_{i=1}^{n} h_{in}\,\Delta_{in}\,.$$

The fundamental lemma may now be restated as follows: *Let $h(x, n)$ be a sequence of horizontal functions on I for $n=1, 2, \cdots$. Let $h(x, n)$ be bounded for all values of n and for*

* Presented to the Society, September 10, 1925.

† *An elementary proof of a fundamental lemma concerning the limit of a sum*, this BULLETIN, vol. 29 (1923), pp. 219–223.