JULIA ON ESSENTIAL SINGULARITIES

Leçons sur les Fonctions Uniformes à Point Singulier Essentiel Isolé. By Gaston Julia. Redigées par P. Flamant. (Borel Monograph.) Paris, Gauthier-Villars, 1924. vii + 149 pp.

This monograph deals with Picard's famous theorem of 1879 to the effect that an analytic function, in the neighborhood of an isolated essential singularity, assumes every value, with two possible exceptions, an infinite number of times. It presents this theorem, and perfections of it due to Landau, Schottky, Caratheodory, Lindelöf, Iversen, and to Julia himself.

A first chapter develops as much of the theory of modular functions as is needed for the proof of Picard's theorem. The theorem that any simply connected region can be mapped conformally upon a circle is here used, and familiarity with it by the reader is assumed. Of course, an elementary proof of Picard's theorem, by the methods of Borel, Landau and Schottky, would not require as much preliminary work, but when one considers the intuitive qualities of Picard's own proof, one can understand Julia's preference. Besides, as Caratheodory has shown, (this is dealt with in the second chapter), the modular function does not play an artificial rôle, for it leads to the determination of an important least upper bound, in the expression for which the modular function actually appears.

The second chapter opens with the proof of Picard's theorem for the special case of an integral function. Then comes the following generalization, due to Landau:

There exists a function $\varphi(a_0, a_1) > 0$, *such that, if the series*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \tag{1}$$

with $a_1 \neq 0$, *converges for* $|x| < \varphi(a_0, a_1)$, *the series assumes at least one of the values* 0 *or* 1 *for* $|x| < \varphi(a_0, a_1)$.

Caratheodory's determination of the best $\varphi(a_0, a_1)$ is then given. As stated above, the modular function appears in this expression. The next theorem is that of Schottky:

There exists a function $M(a_0, \theta)$, *such that, if* (1) *converges for* $|x| \leqq R$ *and does not assume either of the values* 0 *or* 1 *for* $|x| \leqq R$, *we have, for any positive* θ *less than unity, and for* $|x| < \theta R$, *the inequality* $|f(x)| < M(a_0, \theta)$.

The chapter closes with the proof of Picard's theorem for any isolated essential singularity.

The third chapter, one of the most interesting in the book, gives an account of Montel's *normal families* of functions, which Julia used