FUNCTIONS WITH AN ESSENTIAL SINGULARITY*

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1. *Introduction.* In this note we prove certain properties of functions possessing essential singularities. The results grew out of an attempt to prove that the equation

$$\sin x = x$$

has an infinite number of complex roots. This particular fact can be deduced very simply from Picard's theorem (see Theorem VI below), but it suggests other inferences which are much less immediate. The character of the theorems, which are given explicitly below, may be indicated by observing that they prove the existence of an infinite number of roots of the following typical equations:

$$x^3 e^{x^2} + 3x = 0,$$

$$\cos x - \frac{2x+1}{x^2-4} = 0,$$

$$\sec x + 5x = 0.$$

2. *A Corollary to Picard's Theorem.* One possible formulation of Picard's theorem is the following:†

If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is analytic except for a finite number of poles and has only a finite number of zeros, then the equation

$$f(x) - a = 0$$

has an infinite number of roots in the neighborhood in question for all values of a different from zero. The proposed corollary follows.

THEOREM I. *If the function $f(x)$ has an essential singularity at the point P, and in some deleted neighborhood of P is*

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† Cf., e. g., Osgood, *Funktionentheorie*, 2d ed., vol. 1, p. 709.