THREE THEOREMS ON NORMAL ORTHOGONAL SETS

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The three theorems which we desire to record here may be of some interest, in spite of their simplicity. So far as the author knows, they have not appeared in the literature. On the other hand, the corollaries appended are to be found in the journals cited in the footnotes, but were there obtained in different fashion. Throughout our work we shall assume on the part of the reader some familiarity with the theorem of Riesz-Fischer and with convergence in the mean.* All functions considered are understood to be real functions defined on the interval (a, b). They are to be summable with summable square in the Lebesgue sense on this interval. The first result is embodied in the following theorem.

THEOREM I. *A necessary and sufficient condition that every pair of functions f_1, f_2 having identical Fourier coefficients with respect to a normal orthogonal set $\{u_k\}$ should have identical coefficients with respect to a second set $\{v_k\}$ is that the formal series for each function of the set $\{v_k\}$ in terms of $\{u_k\}$ should converge in the mean to that function.*

The condition is necessary. We write

$$\int_a^b v_i u_k = c_{ik},$$

so that the formal series for v_i is $\sum_{k=1}^{\infty} c_{ik} u_k$. Suppose that for some fixed i the series does not converge in the mean

* Plancherel, RENDICONTI DI PALERMO, vol. 30 (1910), pp. 289–335, especially pp. 290–297. See also Riesz, GÖTTINGER NACHRICHTEN, 1907, p. 117.

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