A THEOREM ON SIMPLE ALGEBRAS*

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In a previous paper† I showed that every simple algebra A can be expressed as the direct product of a division algebra D and a simple matric algebra $M = (e_{pq})$; the object of this note is to show that this expression is unique, that is, *if* $A = D_1 \times M_1 = D_2 \times M_2$, *where* D_1 *and* D_2 *are division algebras and* M_1 *and* M_2 *are simple matric algebras, then* D_1 *and* M_1 *are simply isomorphic*‡ *with* D_2 *and* M_2 *respectively.*

Let δ_1 and δ_2 be the orders of D_1 and D_2, and let e_1 and e_2 be primitive idempotent elements of M_1 and M_2 respectively. If e_1 and e_2 are supplementary or equal, then§

$$D_1 \cong e_1 A e_1 \cong e_2 A e_2 \cong D_2\,;$$

M_1 and M_2 are then of the same order and are therefore simply isomorphic. We shall therefore suppose that $e_1 \neq e_2$ and, say, $e_1 e_2 \neq 0$.

Assume in the first place that $x = e_1 e_2$ is not nilpotent; there then exists a rational polynomial

$$y = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_r x^r,$$

which is an idempotent element of the algebra X generated by x. Now $e_1 x = x$; therefore, since every element of X has the form $xf(x)$, $f(x)$ a polynomial in x, it follows that $e_1 y = y$ and

$$(ye_1)^2 = ye_1 ye_1 = y^2 e_1 = ye_1 = e_1 ye_1,$$

so that $ye_1 \leqq e_1 A e_1$; also $ye_1 \neq 0$ since $ye_1 y = y^2 = y \neq 0$; hence ye_1, being idempotent, equals e_1. In the same way

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† PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 6 (1907), p. 99.

‡ *Simple isomorphism* will be denoted by $\cong$.

§ See L. E. Dickson, *Algebras and Their Arithmetics*, Chicago, 1923, pp. 74, 77.