COMPLETE CLASS NUMBER EXPANSIONS FOR CERTAIN ELLIPTIC THETA CONSTANTS OF THE THIRD DEGREE*

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1. *Introduction and Summary.* The expansions of the nine possible theta constants of the third degree in which the parameter of each theta is q,

$$\vartheta_0^3,\ \vartheta_2^3,\ \vartheta_3^3,\ \vartheta_0\vartheta_2^2,\ \vartheta_0\vartheta_3^2,\ \vartheta_2\vartheta_3^2,\ \vartheta_0^2\vartheta_2,\ \vartheta_0^2\vartheta_3,\ \vartheta_2^2\vartheta_3,$$

are of fundamental importance for the arithmetic of class number relations. Two of these. (6), (7) below, have not previously been stated. The expansions of this pair depend upon those of the remaining seven and upon some additional facts concerning representations in a certain ternary quadratic form. The present paper therefore contains the complete set of expansions.

In a recent note† the author has shown that the classical series of Kronecker and Hermite, in which all summations are with respect to $n = 0$ to ∞,

$$(1)\qquad \begin{aligned} \vartheta_2(q^4)\vartheta_3^2(q^4) &= 4\sum q^{4n+1}F(4n+1), \\ \vartheta_2^2(q^4)\vartheta_3(q^4) &= 4\sum q^{4n+2}F(4n+2), \end{aligned}$$

are immediate consequences of the algebraic equivalent

$$(2)\qquad \vartheta_3^3 = 12\sum q^n E(n),\quad E(n) \equiv F(n)—F_1(n),$$

of the theorem of Gauss on representations as sums of three squares.

It was remarked also that their series

$$(3)\qquad \vartheta_2^3(q^4) = 8\sum q^{8n+3}F(8n+3)$$

is also implied by another result due to Gauss. In the foregoing, $F(n)$, $F_1(n)$ denote, respectively, the numbers of

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