ON THE LOCATION OF THE ROOTS OF POLYNOMIALS*

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It is the object of this note to prove a number of results (particularly Theorems I, II, V, VI, below) concerning the roots of polynomials, generalizations of former results established by the writer. Our main new result is the following theorem.

THEOREM I. *Let (the interiors and boundaries of) the circles $C_1, C_2, \ldots, C_k$ whose centers are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the respective loci of $n_1, n_2, \ldots, n_k$ roots of a variable polynomial $f(z)$ which has no other roots, where the circles C_i are all equal and their centers α_i all lie on a line parallel to the axis of reals. If the polynomial*

$$(1)\quad a_0 z^n + n a_1 z^{n-1} + n(n-1) a_2 z^{n-2} + \cdots + n(n-1)\ldots 2\cdot 1\, a_n, \qquad (n = n_1 + n_2 + \cdots + n_k),$$

has only real roots, and if the circles C_i are sufficiently small, then the locus of the roots of the polynomial

$$F(z) = a_0 f(z) + a_1 f'(z) + a_2 f''(z) + \cdots + a_n f^{(n)}(z)$$

consists of the circles C_j' which are equal to the circles C_i and whose centers are the roots of $F(z)$ when the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ of multiplicities $n_1, n_2, \ldots, n_k$ respectively.† *Any circle C_j' which has no point in common with any of the other circles C_m' contains a number of roots of $F(z)$ equal to the multiplicity of its center as a root of $F(z)$ when the roots of $f(z)$ are the points α_i.*

We shall later make clear the exact meaning of the words, "if the circles C_i are sufficiently small". Theorem I is to be proved by iteration of the following theorem.

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† When the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \ldots, \alpha_k$, the roots of $F(z)$ are all collinear with the roots of $f(z)$. This is a well known theorem due to Hermite, which can easily be established by proving the result in succession for the polynomials $F_1(z), F_2(z), \ldots, F_n(z)$ used below.