ANALOGIES BETWEEN THE u_n, v_n OF LUCAS AND ELLIPTIC FUNCTIONS *

BY E. T. BELL

1. *Historical Note.* The u_n, v_n of Lucas are the symmetric functions (n any real integer)

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n,$$

of the roots α, β of $x^2 = px - q$, where p, q are relatively prime integers, so that

$$(u_0, u_1) = (0, 1), \qquad (v_0, v_1) = (2, p),$$

and u_n, v_n are integers satisfying the recurrence

$$x_{n+2} = px_{n+1} - qx_n.$$

The numerous remarkable properties and applications of these integers due to Lucas and others are summarized in vol. 1, chap. XVII of Dickson's *History.* From another source it is known that as early as 1878 Lucas had applied principles similar to those of his fundamental memoir † to symmetric functions of the roots of any algebraic equation and that he had obtained the connection of these, through the intermediary of elliptic and abelian functions, with the theory of numbers. This connection is still to be sought. In 1912 the writer was informed by the late C. A. Laisant, at one time a trustee of Lucas' manuscripts, that there was nowhere in them a vestige of the subject. Nevertheless Laisant recalled vividly that Lucas, about 1878, made a verbal communication to the Société Mathématique de France in which he exhibited a close isomorphism between three symmetric functions, of which one was $\alpha^n + \beta^n + \gamma^n$, of the roots α, β, γ of a cubic equation and the elliptic functions sn, cn, dn, especially as regards a species of double periodicity. All traces of this communica-

* Presented to the Society, September 18, 1923.

† AMERICAN JOURNAL, vol. 1 (1878), p. 184.