DETERMINATION OF ALL SYSTEMS OF ∞^4 CURVES IN SPACE IN WHICH THE SUM OF THE ANGLES OF EVERY TRIANGLE IS TWO RIGHT ANGLES *

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1. *Introduction.* Consider the curves which intersect an arbitrarily chosen system of ∞^1 curves in the plane under a fixed angle α. If α is varied, a system of ∞^2 curves is obtained, termed an *isogonal* family. Isogonal families are characterized by differential equations of the form

$$y'' = (T_x + y'T_y)(1 + y'^2), \tag{1}$$

where T is any function of x and y.

It is easy to prove synthetically that in all isogonal families, and in no other systems of ∞^2 curves in the plane, the sum of the angles of the triangle formed by any three of the curves is equal to π.†

A *natural* family of curves in any space is one obtainable as the system of extremals of a calculus of variations problem of the form

$$\int F ds = \text{minimum}, \tag{2}$$

where F is any point function.‡ In the plane, F is a function of x and y, and the Euler-Lagrange equation of (2) is

$$y'' = (L_y - y'L_x)(1 + y'^2), \tag{3}$$

where $L = \log F$.

Since the family formed by the ∞^2 straight lines of the plane is both isogonal and natural, and since each of these characters is invariant under conformal transformation, every

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† G. Scheffers, *Isogonalkurven, Äquitangentialkurven und komplexe Zahlen*, MATHEMATISCHE ANNALEN, vol. 60 (1905), p. 504.

‡ See E. Kasner, PRINCETON COLLOQUIUM LECTURES (1912), pp. 34–37.