PROOF OF A FORMULA FOR AN AREA*

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1. *Introduction.* The formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)} dudv,$$

representing the area of the image of a (u, v) rectangle R in the (x, y) plane, can be proved to hold in the case where the transformation $x = x(u, v)$, $y = y(u, v)$ is of a very general kind.† The purpose of this paper is to extend and prove the formula by means of an approximating function of a very simple kind. The main properties of this function are given in § 3 and used in the proof of the formula, § 4.

2. *Definitions.* The approximating function for the summable function $f(x, y)$ is given by the formula

$$f^{(\mu)}(x, y) = \frac{1}{\sigma_\mu} \int\int_{\sigma_\mu} f(x + \xi, y + \eta) d\xi d\eta$$

where σ_μ represents the square region $[-\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, also the area of that square.‡ For convenience $f(x, y)$ is regarded as summable (Lebesgue) in the fundamental region S $[0 \leqq x \leqq 1, 0 \leqq y \leqq 1]$ and the properties of $f^{(\mu)}$ are considered with reference to a rectangle R $[a \leqq x \leqq b, c \leqq y \leqq d]$ inside S, μ being less than $a, c, 1 - b, 1 - d$.

The formula to be proved involves *generalized derivatives* and *potential functions.* These are defined as follows:

DEFINITION (i). If α is a given direction and α' the direction 90° in advance of α, the quantity

$$D_\alpha f(x, y) = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_s f(x, y) d\alpha',$$

if it exists, is called the *generalized derivative of f in the direction* α. It is understood that the integral is taken in the positive

* Presented to the Society, December 27, 1922.

† W. H. Young, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 18, p. 339.

‡ It is convenient sometimes to use as σ_μ the interior of the circle $\xi^2 + \eta^2 = \mu^2$. The properties of $f^{(\mu)}$ are essentially the same in this case.